THE COX RING OF A K3 SURFACE WITH PICARD NUMBER TWO

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Abstract. We study generators and relations for Cox rings of K3 surfaces of Picard number 2. The main results are explicit descriptions of the Cox rings when X is a double cover of $\mathbf{P}^2$ ramified over a sextic with a tritangent, a quartic surface in $\mathbf{P}^3$ with a line or a K3 surface with intersection matrix $\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$. This note can be seen as an addendum to a recent paper by Artebani, Hausen and Laface.

1. Introduction

Let X be a smooth projective K3 surface, i.e., a smooth projective surface over an algebraically closed field k with $K_X = 0$ and $H^1(X, \mathscr{O}_X) = 0$. It is of interest to calculate the *Cox ring*, or *total coordinate ring* of X, which is given by

$$\mathrm{Cox}(X) = \bigoplus_{D \in \mathrm{Pic}\, X} H^0\left(X, \mathscr{O}_X(D)\right),$$

where the ring operations are inherited from $k(X)$ by regarding global sections as rational functions. See [HK00] for generalities on Cox rings. It is well-known that the generic K3 surface has Picard number one, and so in this case the Cox ring is isomorphic to the algebra $\bigoplus_{n \geq 0} H^0(X, n\Gamma)$, where Γ is an ample divisor generating $\mathrm{Pic}(X)$. Generators and relations of this ring were investigated by Saint-Donat in [SD74] (see Proposition 2.4)

The next interesting case is when X has Picard number two, so that $\mathrm{Pic}(X)$ is generated by the classes of two effective Cartier divisors Γ_1, Γ_2. The aim in this article is to find an explicit presentation for $\mathrm{Cox}(X)$, i.e find generating sections $x_1, \ldots, x_r$ from the respective vector spaces $H^0(X, D_1), \ldots, H^0(X, D_r)$, and to describe $\mathrm{Cox}(X)$ as a quotient

$$\mathrm{Cox}(X) = k[x_1, \ldots, x_r]/I.$$

Here we consider a $\mathrm{Pic}(X)-$grading on $k[x_1, \ldots, x_r]$ and I given by letting $\deg(x_i) = D_i$.

This paper can be seen as an addendum to the paper [AHL09], where Cox rings of K3 surfaces were first studied. In that paper it was shown that a K3 surface has a finitely generated Cox ring if and only if its effective cone is polyhedral. The authors also look for explicit generators and defining relations for some special K3 surfaces using Laface and Velasco's complex.

The aim of this paper is to present some additional results on the Cox rings of K3 surfaces of Picard number two. Specifically, we classify K3 surfaces whose Cox ring have ≤ 2 defining relations. We also study Cox rings of certain classical K3 surfaces, such as double covers of $\mathbf{P}^2$ ramified over a sextic with a tritangent, quartic surfaces in $\mathbf{P}^3$ with a line or K3 surfaces with intersection matrix $\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$ for $d \geq 2$.

2. Some results on complete linear systems on K3 Surfaces

We recall some standard results on linear systems on K3 surfaces. Most of the results here are due to Saint-Donat [SD74].

Lemma 2.1. [SD74] *Let D be an effective divisor on a K3 surface. Then the linear system $|D|$ has no base-points outside its fixed components. If D is nef, D has a base point if and only if there is a divisor E such that $E^2 = 0$ and $D.E = 1$.*

Date: September 28, 2009.
2000 *Mathematics Subject Classification.* Primary 14J28 Secondary 14M17.
Key words and phrases. Cox rings, total coordinate rings, K3 surfaces.

Lemma 2.2. [SD74, Corollary 2.6] *Let D be a nef divisor class on a K3 surface. If $D^2 > 0$, then $h^1(X, D) = 0$ and the generic member of $|D|$ is smooth and irreducible. Furthermore, if $D^2 = 0$, then $|D|$ is composed with a pencil, i.e $D = rE$, where E is the class of an elliptic curve.*

On a K3 surface, the vanishing of $h^2(X, D)$ for D effective is immediate by duality: $h^2(X, D) = h^0(X, -D) = 0$.

Lemma 2.3. [SD74, Proposition 5.2] *Let D be a nef divisor such that $D^2 \geq 4$. Then D is hyperelliptic only if there exists an elliptic curve E with $D.E = 1$, or $D = 2B$ for some genus 2 curve B.*

Proposition 2.4. [SD74, Theorem 7.2] *Let H be an effective divisor such that $H^2 \geq 8$ such that the general member of $|H|$ is smooth and non-hyperelliptic. Then the algebra $A = \bigoplus_{n\geq 0} H^0(X, nH)$ is generated in degree 1, and the kernel of the map $SymH^0(X, H) \to A$ is generated by elements of degree 2, except if there is a curve E such that $E^2 = 0$ and $E.L = 3$ in which case the ideal is generated in degrees 2 and 3.*

We prove the following result is about the surjectivity of the multiplication maps on a K3 surface.

Proposition 2.5. *Let N be a base-point free divisor and suppose D is an effective divisor such that i) $H^1(D - 2N) = H^1(D - N) = 0$, ii) $D - 3N$ is effective. Then the multiplication map*

$$H^0(X, N) \otimes H^0(X, D - N) \to H^0(X, D)$$

is surjective.

Proof. Let $\mathscr{K}_{0,0}(X, D, N)$ denote the homology of the following complex

$$\bigwedge^{1} H^0(X, N) \otimes H^0(X, D - N) \to \bigwedge^{0} H^0(X, N) \otimes H^0(X, D) \to 0$$

Proving the lemma is equivalent to showing that $\mathscr{K}_{0,0}(X, D, N) = 0$. Now, the assumption i) and the base-point freeness of $|N|$ ensures us that we are in position to apply Green's Duality theorem of [MG84], which states that in these circumstances,

$$\mathscr{K}_{0,0}(X, D, N) \cong \mathscr{K}_{r-2,3}(X, K_X - D, N)^*$$

where $r = h^0(D) - 1$ and $\mathscr{K}_{r-2,3}(X, -D, N)$ is the homology of the complex

$$\begin{aligned} \bigwedge^{r-1} H^0(X, N) \otimes H^0(X, -D + 2N) \quad &\to \quad \bigwedge^{r-2} H^0(X, N) \otimes H^0(X, -D + 3N) \\ &\to \quad \bigwedge^{r-3} H^0(X, N) \otimes H^0(X, -D + 4N). \end{aligned}$$

By assumption $D - 3N$ is effective, hence $H^0(X, -D + 3N) = 0$, and the homology of the complex is zero. This proves the lemma. □

By Theorem 2.11 in [AHL09] the Cox rings are always finitely generated provided that the effective cone is polyhedral. To describe the Cox ring in more detail, we need a good basis for Pic(X). The following result due to Kovacs is a special case of Theorem 2 of [Kov94].

Proposition 2.6. [Kov94] *Let X be a K3 surface with $\rho = 2$. If the effective cone $NE^1(X)$ is polyhedral, it is generated by the classes of curves with self intersection -2 or 0.*

Note that if $NE^1(X) = \langle \Gamma_1, \Gamma_2 \rangle_{\mathbf{R}_{\geq 0}}$ is generated by the classes of Γ_1, Γ_2, then Γ_1, Γ_2 are linearly independent, and hence form a basis for Pic(X). In particular, the proposition implies that we need only consider the cases intersection matrices

$$\begin{pmatrix} -2 & d \\ d & -2 \end{pmatrix}, \quad \begin{pmatrix} -2 & d \\ d & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}. \tag{2.1}$$

where $d = \Gamma_1 \cdot \Gamma_2$. We investigate each of these cases in turn.

3. K3 Surfaces with $\Gamma_1^2 = \Gamma_2^2 = -2$

In this section we consider the case where the Picard group of X is generated by two smooth curves, Γ_1 and Γ_2, with $\Gamma_1^2 = \Gamma_2^2 = -2$. The adjunction formula implies that the Γ_i are rational curves. Let $d = \Gamma_1.\Gamma_2$ be their intersection number. Note that the Hodge index theorem implies that $\begin{vmatrix} \Gamma_1^2 & \Gamma_1\Gamma_2 \\ \Gamma_1\Gamma_2 & \Gamma_2^2 \end{vmatrix} = 4 - d^2 < 0$, so $d \geq 3$. Furthermore $d = 3$ is attainable (see Section 4).

We calculate the effective monoid $\mathrm{Eff}(X)$ and nef monoid $\mathrm{Nef}(X)$ of X. We claim first that $\mathrm{Eff}(X)$ is generated by the classes Γ_1, Γ_2. First of all, it is clear that $\tau = \langle \Gamma_1, \Gamma_2 \rangle_{\mathbf{Z}_{\geq 0}}$ is contained in $\mathrm{Eff}(X)$. Let $\tau^* = \{D \in \mathrm{Pic}(X) | D \cdot \Gamma_i \geq 0, i = 1, 2\}$ denote the dual of τ and note that

$$\begin{aligned} (a\Gamma_1 + b\Gamma_2) \cdot \Gamma_1 \geq 0 \quad &\Longleftrightarrow \quad -2a + db \geq 0 \\ (a\Gamma_1 + b\Gamma_2) \cdot \Gamma_2 \geq 0 \quad &\Longleftrightarrow \quad da - 2b \geq 0. \end{aligned}$$

These inequalities imply that τ^* is generated by the classes

$$(*) \qquad j\Gamma_1 + \Gamma_2, \Gamma_1 + j\Gamma_2 \text{ for } j = 1, \dots, n \text{ if } d = 2n \text{ is even, and}$$

$$(**) \qquad j\Gamma_1 + \Gamma_2,\ \Gamma_1 + j\Gamma_2 \text{ for } j = 1, \dots, n, 2\Gamma_1 + d\Gamma_2 \text{ and } d\Gamma_1 + 2\Gamma_2,\ \text{if } d \text{ is odd.}$$

Note in particular that all of these classes are effective, being non-negative integer combinations of Γ_1, Γ_2. Now, let $D \in \mathrm{Pic}(X)$ be the class of an effective curve. We can write

$$D = n\Gamma_1 + m\Gamma_2 + M$$

where M is an effective divisor with $M \cdot \Gamma_i \geq 0$, i.e $M \in \tau^*$. Since all elements of τ^* are non-negative integer combinations of Γ_1, Γ_2, we have $D \in \tau$. This shows that $\tau = \langle \Gamma_1, \Gamma_2 \rangle_{\mathbf{Z}_{\geq 0}} = \mathrm{Eff}(X)$ and $\mathrm{Nef}(X) = \tau^*$.

We also see that every nef divisor is big, since all the generating classes in the nef monoid have positive self-intersection. In particular, it follows from the Kawamata-Vieweg vanishing theorem and Riemann-Roch that for a nef divisor class $D = a\Gamma_1 + b\Gamma_2$,

$$h^0(X, D) = \frac{D^2}{2} + 2 = dab - a^2 - b^2 + 2. \tag{3.1}$$

Remark. By [BP04, Rem. 1.4] the dimension of the Cox ring is equal to rk $\mathrm{Pic}(X) + \dim(X)$, so the Cox rings is always be four dimensional in this paper.

Proposition 3.1. *Let X be a K3 surface with intersection matrix $\begin{pmatrix} -2 & d \\ d & -2 \end{pmatrix}$. The Cox ring of X is finitely generated, and a minimal generating set of* $\mathrm{Cox}(X)$ *contains sections in classes* Γ_1, Γ_2 *and all the classes* $(*)$ *and* $(**)$*. In particular, such a set must contain at least* $\frac{d(d-2)}{2} + 3$ *elements if d is even, and* $\frac{(d-1)^2}{2} + 4$ *if d is odd.*

Proof. We show first that the sections of all but a finite number nef divisors, can be written as a polynomials in sections of lower degree.

Let $N_1 = 2\Gamma_1 + d\Gamma_2$ and $N_2 = d\Gamma_1 + 2\Gamma_2$ be the extremal rays of the nef cone $NM^1(X)$ and consider the translations

$$\tau_1 = 3N_1 + NM^1(X) \text{ and } \tau_2 = 3N_2 + NM^1(X).$$

By construction, every $D \in \tau_1 \cup \tau_2$ satisfies the assumptions of Lemma 2.5 (with $N = N_1$ or $N = N_2$), and hence the sections of $H^0(X, D)$ are polynomials in sections of lower degree. Since the complement $NM^1(X) \setminus (\tau_1 \cup \tau_2)$ is a finite set of divisors, this shows that the subalgebra $\bigoplus_{D\ nef} H^0(X, D)$ of $\mathrm{Cox}(X)$ is finitely generated.

Let now $D = a\Gamma_1 + b\Gamma_2$ be an arbitrary effective divisor class. If D is not nef, we have say, $l = D \cdot \Gamma_1 < 0$ and an exact sequence

$$0 \to H^0(X, D - \Gamma_1) \to H^0(X, D) \to H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(l)) = 0,$$

and hence all sections of $H^0(X, D)$ are products of sections in $H^0(X, D - \Gamma_1)$ by some section in $H^0(X, \Gamma_1)$, and we may continue the process inductively with $D - \Gamma_1$ until we reach a nef divisor. This shows that $\mathrm{Cox}(X)$ is finitely generated, since $\bigoplus_{D\ nef} H^0(X, D)$ is.

We now look for specific generators. First, we need exactly two generators x, y in degrees Γ_1, Γ_2 respectively. Consider now the classes $D = a\Gamma_1 + \Gamma_2$. These are nef by the previous lemma. Consider now the the multiplication map

$$H^0(X, (a-1)\Gamma_1 + \Gamma_2) \otimes H^0(X, \Gamma_1) \to H^0(X, a\Gamma_1 + \Gamma_2).$$

For $a \leq \lceil \frac{d}{2} \rceil$, this map is never surjective, since $h^0(a\Gamma_1 + \Gamma_2) - h^0((a-1)\Gamma_1 + \Gamma_2) = d - 2a + 1 > 0$. So we need $d - 2a + 1$ new generators in the degrees listed above. Summing the differences gives the above bound on the number of generators. The example in the next section shows that the bound above is attained for $d = 3$. □

Remark. By the bound above, and the fact that $\dim \mathrm{Cox}(X) = 4$, we see that the Cox ring needs ≥ 3 defining relations for $d \geq 4$, and that the number of defining relations grows (at least) quadratically in d.

4. Example: A Double Cover of $\mathbf{P}^2$

Let $\pi : X \to \mathbf{P}^2$ be a double cover of $\mathbf{P}^2$ ramified over a smooth sextic curve $C \subset \mathbf{P}^2$. We assume that there is a line $L \subset \mathbf{P}^2$ which is tritangent to the sextic C. The pullback of L is given by $\pi^*(L) = \Gamma_1 + \Gamma_2$, where Γ_1, Γ_2 correspond to smooth rational curves. By the adjunction formula we have $\Gamma_i^2 = -2$. Also, $(\Gamma_1 + \Gamma_2)^2 = 2L^2 = 2$, since π has degree 2. This gives $d = \Gamma_1 \cdot \Gamma_2 = 3$. Hence X is a K3 surface with intersection matrix

$$\begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}.$$

Let $\sigma : X \to X$ be the involution that switches the sheets of X over $\mathbf{P}^2$. This induces an automorphism $\sigma^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ such that $\sigma^*(\Gamma_1) = \Gamma_2$. This means that $\Gamma = \{\Gamma_1, \Gamma_2\}$ is a nice $\mathbf{Z}$-basis for $\mathrm{Pic}\, X$ and we will use this in the following.

By the above discussion, we have $NE^1(X) = \langle \Gamma_1, \Gamma_2 \rangle_{\mathbf{R}_{\geq 0}}$ and the nef monoid is generated by the divisor classes

$$H := \Gamma_1 + \Gamma_2, \qquad N_1 := 2\Gamma_1 + 3\Gamma_2, \qquad N_2 := 3\Gamma_1 + 2\Gamma_2.$$

This monoid is shown in Figure 1.

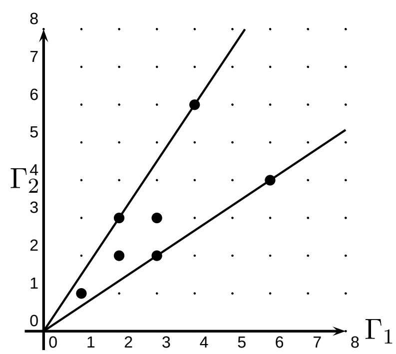

Figure 1. $\mathrm{Nef}(X)$ as a submonoid of $\mathrm{Eff}(X)$ with some special divisor classes plotted.

Lemma 4.1. *Let $X \to B$ be a double cover and let s_R be a section defining the ramification divisor on X. Then*

$$H^0(X, \mathscr{O}_X(kH)) \cong \pi^* H^0(\mathbf{P}^2, \mathscr{O}(k)) \oplus \pi^* H^0(\mathbf{P}^2, \mathscr{O}(k-3)) s_R$$

Proof. This follows from [BPV84, I.17.2] and the projection formula. □

In particular, this implies that we may choose a basis of $H^0(X, H)$ and $H^0(X, 2H)$ consisting of $\sigma-$invariant sections, i.e pullbacks of sections from $\mathscr{O}_{\mathbf{P}^2}(1)$ and $\mathscr{O}_{\mathbf{P}^2}(2)$ respectively.

4.1. **Generators.** We now look for generators for $\mathrm{Cox}(X)$. First, we need at two sections corresponding to the (-2)-curves: We let x, y denote generators for $H^0(X, \Gamma_1)$ and $H^0(X, \Gamma_2)$, respectively.

In degree $H = \Gamma_1 + \Gamma_2$, in which $H^0(X, H)$ is 3-dimensional by Riemann-Roch, we need two new generators z_1, z_2 for a basis, these are invariant under σ by the above remark. Similarly, divisor class $D = 2H$ needs no new sections since $H^0(X, D) \cong \pi^* H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(2)) \cong \mathrm{Sym}^2(H^0(X, H))$.

Consider now the divisor class $D = 2\Gamma_1 + 3\Gamma_3$, which is on the boundary of the nef cone. This has a 7-dimensional cohomology group, while we can only create 6 monomials in degree D with the generators created so far: these are all from the basis for $H^0(X, 2H)$ multiplied with y. Hence we need one new generator for a basis. Call this monomial v.

The same thing happens for the divisor class $D = 3\Gamma_1 + 2\Gamma_2$, and we need another section, say w. We may choose $w = \sigma^*(v)$, which cannot be linearly dependent on monomials in x, y, z_1, z_2, since by applying σ^*, the same would apply to v.

In all, we have shown that we need generators x, y, z_1, z_2, v, w in degrees $\Gamma_1, \Gamma_2, \Gamma_1 + \Gamma_2, 2\Gamma_1 + 3\Gamma_2, 3\Gamma_1 + 2\Gamma_2$ respectively, in accordance with Lemma 5.6. We now claim that these sections are sufficient to generate the Cox ring.

Proposition 4.2. $\mathrm{Cox}(X)$ *is generated by the sections* x, y, z_1, z_2, v, w.

Proof. We use Proposition 2.5 and induction. We first look at some more 'special' divisor classes D, where it is not so obvious that we do not need additional generators. These will also play the role of base cases for the induction.

$D = 3H$. Since $h^0(X, 3H) = 11$ and $\dim_k \mathrm{Sym}^3(H^0(X, H)) = 10$, we need one more section to produce a basis for $H^0(X, 3H)$. Consider the section $v \cdot x$. We claim that this cannot be linearly dependent on the previous monomials. This follows since these are in fact $\sigma-$invariant, while $\sigma(vx) = wy \neq vx$, since $\mathrm{Cox}(X)$ is a UFD. Hence these 11 monomials form a basis for $H^0(X, D)$.

$D = 4\Gamma_1 + 6\Gamma_2$ or $D = 6\Gamma_1 + 4\Gamma_2$. Note that $h^0(X, 4\Gamma_1 + 6\Gamma_2) = 22$ and that $4\Gamma_1 + 6\Gamma_2 = (4\Gamma_1 + 5\Gamma_2) + \Gamma_2$. Consider the divisor $D' = 4\Gamma_1 + 5\Gamma_2$. D' is nef and big since $D' = (2\Gamma_1 + 3\Gamma_2) + 2(\Gamma_1 + \Gamma_2)$ and has $h^0(X, D') = 21$. Hence by multiplying a base of $H^0(X, D')$ by y we get 21 linearly independent sections in $H^0(X, D)$. Now we add the section v^2, which cannot be a linear combination of the other monomials since these are all divisible by y. By switching the roles of Γ_1 and Γ_2 we also prove it for the divisor $6\Gamma_1 + 4\Gamma_2$.

So far, all we have done is given generators for $\mathrm{Cox}(X)_D$ for D plotted in Figure 1. We now show that the sections above are sufficient to generate the whole ring, and proceed with induction on the number $\kappa = a + b \geq 0$. Let $D = a\Gamma_1 + b\Gamma_2$ be an effective divisor class. As before we may assume D to be nef (and big).

Case 1: $a = b = n$. Here $D = nH$. We have already considered the cases $n \leq 3$. Suppose that $n \geq 4$. We now recall a theorem from classical curve theory:

Lemma 4.3 (Noether's theorem)**.** *Let C be a smooth curve of genus g and let $R_C = \bigoplus_{n \geq 0} H^0(C, nK_C)$ be its canonical ring.*

1. *If C is not hyperelliptic, the R_C is generated in degree 1.*
2. *If $g = 2$, and C is hyperelliptic, then R_C is generated by elements of degree 1, and by 1 element of degree 3.*
3. *If $g \geq 3$, and C is hyperelliptic, then R_C is generated by elements of degree 1, and by $g - 2$ elements of degree 2.*

Let C be an irreducible curve in $|H|$. Note that we have the exact sequence

$$0 \to H^0(X, (n-1)C) \to H^0(X, nC) \to H^0(C, nK|_C) \to 0.$$

since $H^0(C, nH|_C) = H^0(C, nK|_C)$ by adjunction. Now, since $H^0(X, 3H) \to H^0(C, 3K|_C)$ is surjective, we choose a set of sections from $H^0(X, 3H)$ mapping isomorphically to a basis for $H^0(C, 3K|_C)$. By the above lemma the elements of $H^0(C, nK|_H)$ are polynomials in sections from $H^0(C, K|_H)$ and $H^0(C, 3K|_H)$. By the above exact sequence, it follows that sections in $H^0(X, nH)$ for $n \geq 4$ are polynomials in sections of lower degree.

Case 2: $a > b$. Write for simplicity $N = 3\Gamma_1 + 2\Gamma_2$, and note that in this case (where $a > b$), D can be written uniquely in the from

$$D = mN + nH \quad m \geq 1, n \geq 0$$

We use induction on n. If $n = 0$, we choose an irreducible curve $C \in |N|$. By the adjunction formula, C has genus 6, and it follows from Lemma 4.3 that the algebra $\bigoplus_{r\geq 0} H^0(C, rK_C)$ is generated in degrees ≤ 2. We proceed as before and use the exact sequence

$$0 \to H^0(X, (m-1)N) \to H^0(X, mN) \to H^0(C, mN|_C) \to 0..$$

to conclude that sections in $H^0(X, mN)$ are polynomials in sections of lower degree, for all $m \geq 3$. Hence the result follows by induction on m.

If $n = 1$, then $D = mN + H$. We check the assumptions of Proposition 2.5 to ensure that we have a surjection $H^0(X, H) \otimes H^0(X, D - H) \to H^0(X, D)$, then the result will follow by induction on κ.

First, $H^1(X, D - H) = H^1(X, mN) = 0$ by nef and bigness of N. Now, $D - 2H$ is not nef (it has Γ_1 as a fixed component), but we will verify that $H^1(X, D - 2H) = 0$. We have that $D - 2H = mN - (\Gamma_1 + \Gamma_2) = (m - 1)N + 2\Gamma_1 + \Gamma_2$, and that the long exact sequence of cohomology applied to the sequence

$$0 \to \mathscr{O}_X(D - 2H - \Gamma_1) \to \mathscr{O}_X(D - 2H) \to \mathscr{O}_{\Gamma_1}(D - 2H) \to 0$$

gives

$$0 \to H^1(X, (m-1)N + \Gamma_1 + \Gamma_2) \to H^1(X, D - 2H) \to H^1(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(-1)) \to 0$$

and so $H^1(X, D - 2H) = 0$ by exactness.

If $n \geq 2$, then both $D - H$ and $D - 2H$ are nef and big, so the criteria of Proposition 2.5 are satisfied.

Case 3: $a < b$. The argument is completely analogous to that of Case 2, by switching the roles of Γ_1 and Γ_2. □

Remark. This approach can with some modification be used in tackling K3 surfaces with intersection matrix $\begin{pmatrix} -2 & d \\ d & -2 \end{pmatrix}$ for higher d.

4.2. **Relations.** We now describe the relations among the generators x, y, z_1, z_2, v, w in $\mathrm{Cox}(X)$. By using the Reynolds operator and Noether's theorem, we find that the polynomials invariant under σ are exactly the polynomials in z_1, z_2, xy, $vx + wy$ and vw, i.e

$$k[x, y, z_1, z_2, v, w]^{\langle\sigma\rangle} = k[z_1, z_2, xy, vx + wy, vw].$$

Consider the the expression $xv + yw$. Since it is invariant under σ, we can write it as a pullback of a section from $H^0(\mathbf{P}^2, \mathscr{O}(3))$, that is, in terms of z_1, z_2, xy, and hence we have a relation of the form

$$f := xv + yw - \alpha(xy, z_1, z_2) = 0.$$

where α is a homogeneous degree 3 polynomial. Now there are exactly 12 monomials in $k[x, y, z_1, z_2, v]$ of degree $D = 3H$: 10 of these come from $\mathrm{Sym}^3(H^0(X, H))$, and we have in addition the sections xv, yw. Since $h^0(X, D) = 11$, this shows g is the only relation in degree D and so $I_D = (f)_D$.

Similarly, vw is an σ-invariant section of degree $5H$ in $\mathrm{Cox}(X)$ and thus it can be written as a pullback of a sections from $H^0(\mathbf{P}^2, \mathscr{O}(5))$. That means we have a relation of the form

$$g := vw - \beta(xy, z_1, z_2) = 0$$

where β has degree 5.

Now, there are 34 monomials in degree $D = 5H$, and $h^0(X, 5H) = 27$ and so $\dim I_{5H} = 7$. Note that since there are 6 monomials in $k[x, y, z_1, z_2, v]_{2H}$, so $\dim_k(f)_{5H} = 6$ (since f has degree $3H$), and hence there is exactly one minimal relation of degree $5H$, namely g. This shows that $I_{5H} = (f, g)_{5H}$:

We denote the ideal generated by f and g by J. We now claim that $\mathrm{Cox}(X) \cong k[x, y, z_1, z_2, v]/J$, which is reasonable as the two rings have the same dimension.

Lemma 4.4. *The elimination ideal $k[x,y,z_1,z_2,v]\cap J$ is generated by $h = yf - vg$, the resultant of f and g with respect to the variable w. Mutatis mutandis for the ideal $k[x,y,z_1,z_2,w]\cap J$.*

Proof. Write $R = k[x,y,z_1,z_2,v]$. Note that $h \in R\cap J$, while it is not so clear that it is a generator for the elimination ideal. However, let $P = pf - qg$ be an arbitrary element in $R\cap J$, where $p = \sum_{k=0}^n a_k w^k$, $q = \sum_{k=0}^n b_k w^k$ are considered as elements in $R[w]$.

Claim: We may assume $n = 0$. Suppose $n > 0$. Since the terms in P involving w^{n+1} must cancel we must have $a_n v = b_n y$, and consequently there is an $r \in R$ such that $a_n = yr$ and $b_n = vr$. Hence

$$P = pf - qg = (p - rgw^{n-1})f - (q - rfw^{n-1})$$

Now $p - rgw^{n-1} = (a_{n-1} - xvr + \beta r)w^{n-1} + \ldots$, and $q - rfw^{n-1} = (b_{n-1} - r\alpha)w^{n-1} + \ldots$ are polynomials in w of degrees $< n$, so by iterating this process, we eliminate successive powers of n.

For $n = 0$, the claim is immediate, since

$$P = pf - qg = w(pv - qy) + \text{ ”terms not containing } w\text{”}$$

Since $pv - qy$ must vanish, there is an $r \in R$ such that $p = yr, q = vr$, hence

$$P = yr \cdot f - vr \cdot g = r(yf - vg) \in (h).$$

□

Theorem 4.5. *Let X be a K3 surface with intersection matrix $\begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}$. Then the Cox ring* $\mathrm{Cox}(X)$ *is a complete intersection ring isomorphic to a quotient of $k[x,y,z_1,z_2,v,w]$ by the ideal $J = (f,g)$, where*

$$f = xv + yw - \alpha(xy, z_1, z_2), \qquad g = vw - \beta(xy, z_1, z_2)$$

where α, β are polynomials of degrees 3 and 5 respectively.

Proof. By Proposition 4.2 we have a surjection

$$k[x,y,z_1,z_2,v,w]/J \to \mathrm{Cox}(X).$$

We show that this is an isomorphism in all degrees corresponding to nef divisor classes $D = a\Gamma_1 + b\Gamma_2$, that is, we show that

$$\dim_k \left(k[x,y,z_1,z_2,v,w]/J\right)_D = \dim_k \mathrm{Cox}(X)_D = 3ab - a^2 - b^2 + 2.$$

Note that using the relations f and g we may modulo J eliminate all terms containing vw and yw, and hence we may decompose $k[x,y,z_1,z_2,v,w]/J$ as a vector space

$$k[x,y,z_1,z_2,v,w]/J \cong \bigoplus_{n>0} k[x,z_1,z_2]w^n \oplus k[x,y,z_1,z_2,v]/(h)$$

where $h = yf - vg$ is the generator for the elimination ideal $J' = k[x,y,z_1,z_2,v]\cap J$. Our job is now to calculate the dimensions of these two vector spaces in degree D separately. We may assume for the moment that $a \geq b$. The case where $a \leq b$ is completely analogous, and is obtained by switching the roles of v and w above.

$\dim_k \left(\bigoplus_{n>0} k[x,z_1,z_2]w^n\right)_D$. Note that we are looking for the number of monomials m in $k[x,z_1,z_2]$ such that $\deg mw^k = a\Gamma_1 + b\Gamma_2$ for some $k \in \mathbf{N}$. By looking at these monomials' degrees, we find that this problem is equivalent to the following counting problem: Find the number of non-negative integer solutions to the system

$$\begin{aligned} a_1 + \quad a_2 + a_3 + 3a_4 &= a \\ a_2 + a_3 + 2a_4 &= b \end{aligned} \tag{4.1}$$

Write this as

$$\begin{aligned} a_1 + a_2 + a_3 &= a - 3a_4 \\ a_2 + a_3 &= b - 2a_4 \end{aligned} \tag{4.2}$$

and note that given a solution to the 2nd equation uniquely determines a_1 as $a_1 = a - b - a_4$. Of course, $a - b \geq 0$ by assumption, so its clear that we must restrict ourselves to values of a_4 in the range $0 \leq a_4 \leq a - b$ to ensure non-negativity of a_1. Note that in this case we have

$$b - 2a_4 \geq b - 2(a - b) = 3b - 2a = D \cdot \Gamma_1 \geq 0$$

where the last inequality is precisely ensured by the nef condition on D (!). Now we find the number of solutions to (4.2) by counting: for every $0 \leq a_4 \leq a - b$, we seek the number of ways of writing $b - 2a_4$ as a sum of two non-negative integers a_2, a_3, which is $b - 2a_4 + 1$, hence the total number of solutions to the system is given by

$$\begin{aligned} \sum_{i=1}^{a-b} (b - 2i + 1) &= b(a - b) - (a - b + 1)(a - b) + (a - b) \\ &= 3ab - 2b^2 - a^2. \end{aligned}$$

$\dim_k (k[y, z_1, z_2, v]/(h))_D$. Write $S = k[y, z_1, z_2, v]$ and let $\chi(a, b)$ be the number of monomials in $S_{a\Gamma_1 + b\Gamma_2}$. Note that h has degree $5\Gamma_1 + 6f_2$. By the exact sequence

$$0 \to S(-5\Gamma_1 - 6\Gamma_2) \to S \to S/(h) \to 0$$

we get $\dim_k (S/(h))_D = \chi(a, b) - \chi(a - 5, b - 6)$. As before the dimension count reduces to the combinatorial problem of finding the number of solutions $\chi(a, b)$, to

$$\begin{aligned} a_1 + a_3 + a_4 + 2a_5 &= a \\ a_2 + a_3 + a_4 + 3a_5 &= b \end{aligned} \tag{4.3}$$

and our goal is to get an expression for $\chi(a, b) - \chi(a - 5, b - 6)$. Note that any solution to the 2nd equation gives a_1 uniquely determined as $a_1 = a - b + a_2 + a_5 \geq 0$, hence as long as $a \geq b$, we need only find the number of solutions to the 2nd equation. Of course, the number of non-negative integer solutions to $a_2 + a_3 + a_4 + 3a_5 = b$ appears as the coefficient of x^b in the expression $(1 + x + x^2 + \ldots)^3 \cdot (1 + x^3 + x^6 + \ldots) = \frac{1}{(1-x)^3(1-x^3)}$. Hence $\chi(a, b) - \chi(a - 5, b - 6)$ is equal to the coefficient of x^b in the following expression:

$$\begin{aligned} \frac{1 - x^6}{(1 - x)^3(1 - x^3)} &= \frac{1 + x^3}{(1 - x)^3} \\ &= 1 + \sum_{n=1}^{\infty} \left(\binom{n+2}{2} + \binom{n-1}{2} \right) x^n \\ &= 1 + \sum_{n=1}^{\infty} (n^2 + 2)x^n \end{aligned}$$

This shows that $\chi(a, b) - \chi(a - 5, b - 6) = b^2 + 2$

In all we have that

$$\begin{aligned} \dim_k (k[x, y, z_1, z_2, v, w]/J)_D &= (3ab - 2b^2 - a^2) + (b^2 + 2) \\ &= 3ab - a^2 - b^2 + 2 = h^0(X, a\Gamma_1 + b\Gamma_2). \end{aligned}$$

This finishes the proof. □

Remark. This example coincides with Example 5.8 in [AHL09], but the authors do not prove sufficiency of the generators nor the minimal generators for the ideal.

5. K3 surfaces with $\Gamma_1^2 = -2$ and $\Gamma_2^2 = 0$

In this section we consider the case where $\mathrm{Pic}(X)$ is generated by the classes of two curves, say, Γ_1 and Γ_2, where $\Gamma_1^2 = -2$ and $\Gamma_2^2 = 0$. Let $d = \Gamma_1 \cdot \Gamma_2$ be the number of points they intersect taken with multiplicity. The intersection matrix is then given by

$$\begin{pmatrix} -2 & d \\ d & 0 \end{pmatrix}$$

Lemma 5.1. *The effective cone $NE^1(X)$ is generated by Γ_1 and Γ_2. Also, these generate the monoid of effective divisor classes* $\mathrm{Eff}(X, \mathbf{Z})$.

If $d = 2n$ the nef monoid is generated by the classes $a\Gamma_1 + \Gamma_2$ for $a = 1, \dots, n$ and if $d = 2n+1$ we need also the divisor class $d\Gamma_1 + 2\Gamma_2$.

Theorem 5.2. *The Cox ring of X is finitely generated, and any generating set of sections contains at least $\frac{d^2}{4} + 3$ elements if d is even and $\frac{d^2-1}{4} + 4$ elements if d is odd.*

The proofs of these results are similar to the proof of Proposition 3.1 and the discussion preceding and is therefore omitted.

5.1. Results for low d.

For low d the Cox ring needs few relations, and it is possible to calculate the ideal of relations. The results are summarized in the following theorem.

Theorem 5.3. *Let X_d be any smooth projective K3 surface with intersection matrix* $\begin{pmatrix} -2 & d \\ d & 0 \end{pmatrix}$.

- $d = 1$. *The Cox ring of X_1 is*
$$\mathrm{Cox}(X_1) = k[x, y_1, y_2, z, t]/(f)$$
where $\deg(x) = \Gamma_1, \deg(y_i) = \Gamma_2, \deg(z) = 2\Gamma_1 + 4\Gamma_2, \deg(z) = 3\Gamma_1 + 6\Gamma_2$ *and* $\deg(f) = 6\Gamma_1 + 12\Gamma_2$.
- $d = 2$. *The Cox ring of X_2 is*
$$\mathrm{Cox}(X_2) = k[x, y_1, y_2, z, t]/(g)$$
where $\deg(x) = \Gamma_1, \deg(y_i) = \Gamma_2, \deg(z) = \Gamma_1 + \Gamma_2, \deg(z) = 2\Gamma_1 + 3\Gamma_2$ *and* $\deg(g) = 4\Gamma_1 + 6\Gamma_2$.
- $d = 3$. *Then the Cox ring of X_3 is a complete intersection ring*
$$k[x, y_1, y_2, z_1, z_2, t]/(f, g)$$
where $\deg(z) = \Gamma_1, \deg y_i = \Gamma_2, \deg z_i = \Gamma_1 + \Gamma_2, \deg t = 3\Gamma_1 + 2\Gamma_2$. *The ideal is generated by two relations f, g of degree* $3\Gamma_1 + 3\Gamma_2$.

For $d \geq 4$, the number of defining relations in $\mathrm{Cox}(X)$ *is at least* 3.

Proof. Since the method of computation is similar to that of the K3 surface in Section 4.2, we provide only a sketch of the proof. As before we find generators of $\mathrm{Cox}(X)$ by looking in low degree nef classes. For example, we need a section $x \in H^0(X, \Gamma_1)$, and two basis elements y_1, y_2 from $H^0(X, \Gamma_2)$.

The rest is a direct checking using the fact that Γ_2 moves in a pencil and the base-point free pencil trick, and lemma 2.5 for the remaining divisors.

Concerning the relations, Note that $\mathrm{Cox}(X_1)$ and $\mathrm{Cox}(X_2)$ are hypersurfaces since they are minimally generated by 5 sections and $\mathrm{Cox}(X)$ is of dimension four. Summarily, X_3 will require requires two minimal relations. These occur in degree $3\Gamma_1 + 3\Gamma_2$, since $h^0(3\Gamma_1 + 3\Gamma_2)20$ and there are exactly 22 monomials of this degree: 20 forming a basis for $\mathrm{Sym}^3(H^0(X, \Gamma_1 + \Gamma_2))$ plus the monomials ty_1, ty_2. Hence we have two relations of the form

$$ty_i = f_i(x, y_1, y_2, z_1, z_2)$$

The argument to show that these relations generate the ideal is done by a (slightly less complicated) combinatorial argument as in Theorem 4.5. □

Remark. Surfaces of type X_3 can be realized as quartic surfaces X in $\mathbf{P}^3$ containing a line. The Cox ring of such a surface was also studied in [GM00], where the authors refer to it as the *Mori quartic*.

6. K3 Surfaces with $\Gamma_1^2 = \Gamma_2^2 = 0$

Consider the case where the surface X has intersection matrix $\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$, which means that the Picard group $\mathrm{Pic}(X)$ is generated by classes of smooth elliptic curves Γ_1 and Γ_2. Cox rings of K3 surfaces of this type were considered in [AHL09], where the authors also use Laface and Velasco's complex to find the relations.

The divisor classes of Γ_1, Γ_2 generate $NE^1(X)$ and since $\Gamma_i \cdot \Gamma_j \geq 0$ for $1 \leq i, j \leq 2$, this shows that also the effective cone is generated by these curves and equals the nef cone in this basis. So every effective divisor $a\Gamma_1 + b\Gamma_2$ is effective/nef iff $a, b \geq 0$, and ample as long as $a, b \geq 1$. In this case the Riemann-Roch theorem gives the following formula:

$$h^0(X, a\Gamma_1 + b\Gamma_2) = \frac{1}{2}(a\Gamma_1 + b\Gamma_2)^2 + 2 = abd + 2.$$

Write again for simplicity $H = \Gamma_1 + \Gamma_2$. Note that H is an ample divisor on X, and that $H^2 = 2d$.

Note that since $h^0(\Gamma_1) = 2$, we need two generators x_1, x_2 for $H^0(X, \Gamma_1)$, and similarly two generators y_1, y_2 for $H^0(X, \Gamma_2)$.

Proposition 6.1. *Let X_d be a K3 surface with intersection matrix* $\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$.

- *If $d = 2$, $\mathrm{Cox}(X)$ is generated by x_1, x_2, y_1, y_2, z where $\deg z = 2H$.*
- *If $d \geq 3$, $\mathrm{Cox}(X)$ is generated by $x_1, x_2, y_1, y_2, z_1, \dots, z_{d-2}$, where $\deg z_i = H$.*

Proof. We have that $h^0(X, H) = d+2$, so we need $d-2$ new generators in degree H. Let $D = a\Gamma_1 + b\Gamma_2$ be an effective (hence nef) divisor class. We may suppose $a \geq b$. Note that the Γ_i are basepoint free pencils, so the base-point free pencil trick gives us a surjection

$$H^0(X, D - \Gamma_1) \otimes H^0(X, \Gamma_1) \to H^0(X, D)$$

provided that $H^1(D - 2\Gamma_1) = 0$, which is the case for all divisors $D = a\Gamma_1 + b\Gamma_2$ with $a > 2$ or $(a, b) = (2, 1), (1, 2)$. It follows that we reduce to checking degrees H and $2H$. We apply the trick from before by using Noether's theorem

$$0 \to H^0(X, H) \to H^0(X, 2H) \to H^0(C, 2K|_C) \to 0$$

If $d \geq 3$ the claim follows now by reasoning as in the proof of Proposition 4.2 since H is non-hyperelliptic for $d \geq 3$ (by Lemma 2.3) and so $\bigoplus_{n \geq 0} H^0(X, nK_C)$ is generated in degree H. If $d = 2$, H is hyperelliptic and we need a section in degree $2H$. □

Remark. A double cover of $\mathbf{P}^1 \times \mathbf{P}^1$ ramified over a curve of bidegree $(4, 4)$ gives an example of a K3 surface with intersection matrix $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

6.1. Relations.

It is possible to say something about the relations in the Cox ring for every d in this case. This is much owed to the facts that Γ_1, Γ_2 move in pencils, and the exact sequence

$$0 \to H^0(X, D - 2\Gamma_1) \xrightarrow{a} H^0(X, D - \Gamma_1) \otimes H^0(X, \Gamma_1) \xrightarrow{b} H^0(X, D) \to H^1(X, D - 2\Gamma_1).$$

where the last H^1 is zero for $D - 2\Gamma_1$ nef and big. The maps here are as follows: $a(s) = sx_1 \otimes x_2 - sx_2 \otimes x_1$ and $b(t \otimes x_i) = tx_i$. The first thing to note is that all monomials of degree $nH = n\Gamma_1 + n\Gamma_2$ are divisible by x_1 or x_2 except the ones that are products of z_i's. As it turns out, this easy observation and the above sequence will be sufficient for proving that the ideal of relations is generated in degree $2H$.

Theorem 6.2. *Let X be a a K3 surface with intersection matrix* $\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$ *and let H be the class of $\Gamma_1 + \Gamma_2$. If $d = 2$, the Cox ring is isomorphic to a quotient $k[x_1, x_2, y_1, y_2, z]/(z^2 - F)$, where F is a polynomial of degree $4H$.*

If $d \geq 3$, the Cox ring is a quotient

$$k[x_1, x_2, y_1, y_2, z_1, \dots, z_{d-2}]/I,$$

where the ideal is generated by $\binom{d-1}{2} - 1$ relations of degree $2H$ if $d \geq 4$ and one relation if $d = 3$.

Proof. Suppose first that $d \geq 3$. First we claim that there are $\binom{d-1}{2} - 1$ quadrics in I_{2H}. Riemann-Roch gives $h^0(X, 2H) = 4d + 2$. Now the monomials $z_i z_j$ give $\binom{d-1}{2}$ monomials in degree $2H$, and we need $4 \cdot (d-2)$ monomials of the form $x_i y_j z_j$ and $x_i x_j y_k y_l$ give 9 monomials. In all

$$\dim_k I_{2H} = \binom{d-1}{2} + 4 \cdot (d-2) + 9 - (4d+2) = \binom{d-1}{2} - 1.$$

Note that any non-trivial relation $f \in I_{2H}$ must involve some $z_i z_j$ terms, since otherwise we may write $f = x_1 P + x_2 Q = 0$, hence by the UFD property, f is a product of $x_1 x_2$ times a relation of degree $2\Gamma_2$, a contradiction.

When $d = 3$, by the above reasoning, we have a single relation of the form $z^2 = Px_1 + Qx_2$ where $P, Q \in k[x_1, x_2, y_1, y_2, z]$.

For $d \geq 4$, note that the number of monomials $z_i z_j$ is exactly one more than the number of relations. It follows we have minimal relations of the form

$$z_i z_j = P_{ij} x_1 + x_2 Q_{ij} + c_{ij} z_m z_n \qquad P_{ij}, Q_{ij}, c_{ij} \in k[x_i, y_i, z_i], \text{ for all } i \neq j \tag{6.1}$$

for some fixed $1 \leq m, n \leq d-2$. Denote their ideal by J.

Let $D = a\Gamma_1 + b\Gamma_2$ is the class of an effective divisor and suppose $a \geq b$. We induct on the number $\kappa = a + b$. Let $A = R/J$ and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\psi \arrow[r] \arrow[d] & A_{D-\Gamma_1} \otimes A_{\Gamma_1} \arrow[r, "\psi"] \arrow[d, equal] & A_D \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & H^0(X, D - 2\Gamma_1) \arrow[r] & H^0(X, D - \Gamma_1) \otimes H^0(X, \Gamma_1) \arrow[r] & H^0(X, D) \arrow[r] & 0
\end{tikzcd}$$

where the middle vertical map is an isomorphism by induction on κ, and the bottom sequence is exact. We claim that the middle sequence is also exact, i.e

Claim: The map $\psi : A_{D-\Gamma_1} \otimes A_{\Gamma_1} \to A_D$ is surjective.

To see why this implies the result, note that $A_{D-2\Gamma_1} \subseteq \ker\psi$ maps surjectively to $H^0(X, D - 2\Gamma_1)$. By the snake lemma and exactness we have that $\ker p = 0$, and so $A_D \cong H^0(X, D)$.

Proof of Claim: We show that we may modulo the relations (6.1) write any monomial as a sum of terms divisible by either x_1 or x_2. For $d = 3$, this is immediate since we may use the relation $z^2 - F$ to reduce the monomial $x_1^{i_1} x_1^{i_2} y_1^{j_1} y_1^{j_2} z^n$ to a linear combination of terms of degree at most one in z - by the multigrading these terms must be divisible by either x_1 or x_2.

For $d \geq 4$, the same argument and the equations (6.1) are *almost* enough to ensure the surjection. We need more information about the relations. We first use Proposition 2.4 to conclude the defining ideal of the algebra $\bigoplus_{r \geq 0} H^0(X, rH)$ is generated by quadrics if $d \geq 4$. In particular, there are no minimal relations in degree $3H$, and so $A_{3H} = H^0(X, 3H)$. Consider then the diagram

$$\begin{tikzcd}
0 \arrow[r] & A_{3H-2\Gamma_1} \arrow[r] \arrow[d, equal] & A_{3H-\Gamma_1} \otimes A_{\Gamma_1} \arrow[r, "\psi"] \arrow[d, equal] & A_{3H} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^0(X, 3H - 2\Gamma_1) \arrow[r] & H^0(X, 3H - \Gamma_1) \otimes H^0(X, \Gamma_1) \arrow[r] & H^0(X, 3H) \arrow[r] & 0
\end{tikzcd}$$

Since the bottom right map is surjective and $z_i z_j z_k \in A_{3H}$ we have that $z_i z_j z_k \in x_1 R_{3H-\Gamma_1} + x_2 R_{3H-\Gamma_1}$ modulo J for all $1 \leq i, j, k \leq d-2$.

Now the surjection $A_{D-\Gamma_1} \oplus A_{D-\Gamma_1} \to A_D$ is clear. Indeed, if $D \neq nH$, then any monomial of degree D must be divisible by x_1 or x_2 (since the z_i all have degree H) and the map is surjective. Now, if $D = nH$, and $n \geq 3$, then a monomial $z_1^{n_1} \cdots z_{d-2}^{n_{d-2}}$ may, by chopping off three z_i's in an arbitrary manner, be written as a linear combinations of terms divisible by x_1 or x_2 modulo the relations above. This proves the theorem for $d \geq 3$.

If $d = 2$, By Riemann-Roch, $h^0(X, 4H) = 34$, while there are 35 monomials of degree $4H$: z^2 and 34 monomials from $\operatorname{Sym}^2 H^0(X, 2H)$. This means that we have a relation in degree $4H$ $z^2 - F$ where $F \in R$ is a polynomial of degree $4H$. Notice that all the terms of the polynomial F must have x_i's in

them (z^2 is the only term of degree $4H$ without x_1 or x_2). This means that $F = x_1 f + x_2 g$ and we may use the relation $z^2 - F$ and the argument above to get a surjection $A_{D-\Gamma_1} \otimes A_{\Gamma_1} \to A_D$. $\square$

Remark. Note that the defining relations above come from the kernel of the map $\mathrm{Sym} H^0(X,H) \to \bigoplus_{n\geq 0} H^0(X,nH)$, i.e the relations defining the K3 surface. By Proposition 2.4 this is generated in degree 2 by precisely $\binom{d-1}{2}$ quadrics. Going to the the Cox ring, we loose one relation, namely $(x_1y_1)(x_2y_2) - (x_2y_1)(x_1y_2)$.

Acknowledgement. The results here were obtained in my master thesis at the University of Oslo. I wish to thank my thesis supervisor Kristian Ranestad for introducing this problem to me and for continuous encouragement. I would also like to thank Antonio Laface for helpful correspondence.

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